LEARNERS' EXPLANATIONS OF THE ERRORS THEY MAKE IN INTRODUCTORY ALGEBRA

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This paper explores the thinking underlying Grade 9 learner errors in introductory algebra. The research used qualitative methods composed of documentary analysis as well as task-based interviews. Data was analysed using Kuchemann's (1981) six interpretations of letters in algebra. The results indicate that various aspects contribute to learners' errors including task instructions, new knowledge, ignoring of the letter, and replacing letters with numeric values.

INTRODUCTION

I begin this paper by giving the background to the study. I then briefly provide the conceptual framework and literature review about the research study. Thereafter I elaborate on the methods used as well as the participants in the study. I then describe how the data was collected. This is then followed by data analysis and discussion of findings.

BACKGROUND TO THE STUDY

Prior to this research project, I was ignorant of the errors and misconceptions that learners produce in algebra and could not address these in my teaching. The problem of low learner achievement in South Africa does not seem to be subsiding (Simkins, 2013). Moreover, the Annual National Assessment (ANA) of 2011 revealed that, "the overall performance of learners was very low with average scores of 30%" (DBE, 2011, p. 2). These results also indicated that "domains in which learners displayed most serious weaknesses included patterns and mathematical functions" (DBE, 2011, p. 33). In addition, poor performance in higher grades (9-12) was linked to poor performance in algebra.

According to the RNCS "algebra is the language for investigating and communicating most of Mathematics and it can be seen as generalised arithmetic, extended to the study of functions and other relationships between variables" (DoE, 2002, p. 62). Algebra plays an important role in high school and tertiary education. Understanding why learners produce errors in algebra might be the beginning of the solution to low performance in mathematics since it may enable teachers to identify difficulties and obstacles that learners encounter when developing algebraic concepts. Hence, attention needs to be given to helping teachers to teach algebra in a meaningful way where they would address and rectify the usual errors that learners produce. I believe that learners and our education system in general stand to benefit from the findings of this research.

PURPOSE OF THE RESEARCH

The main purpose of the research project was to explore learner thinking underlying errors that learners produce in introductory algebra. This was done by looking into how Grade 9 learners interpret letters in different algebraic tasks. The research project was guided by the following question:

- What are learners' explanations behind their responses to introductory algebraic tasks?

LITERATURE REVIEW

According to the Oxford dictionary, algebra is a part of mathematics that uses letters and other symbols to represent quantities and situations. Furthermore, algebra is one of the most important topics in mathematics that develop learners' problem solving and analytical thinking skills (Schoenfeld, 2007). Hence, learner thinking is of great importance in this study. I would define learner thinking as the process that occurs in learner's mind when applying the existing knowledge to solve a certain task. According to Olivier (1989), the existing knowledge is structured in a learner's mind into interrelated concepts called schema. These schemas are important tools that can be retrieved and used by learners when they encounter a similar scenario. If the new knowledge is not connected to the existing schema, that knowledge becomes isolated, which results in learners memorizing it in a form of rote learning (Olivier, 1989). Hence, "errors and misconception are the natural results of learners' effort to construct their own knowledge" (Olivier, 1989, p. 13).

Large-scale research in countries such as Britain (Kuchemann, 1981), Australia (Stacey & MacGregor, 1994 & 1997) and others has shown that most learners encounter learning difficulties in understanding algebra and this has manifested in the range of errors and misconceptions, which are a characteristic of learners' responses. According to Christou, Vosniadou & Vamvakossi, the tendency of learners "to use their prior experience with numbers in the context of arithmetic" (2007, p. 289) interferes with the interpretation of letters in algebra. Again, learners have a tendency of simplifying an algebraic answer to a single term. Booth (1999) and Stacey & MacGregor (1994) refer to this error as conjoining of algebraic terms. I have witnessed these kinds of errors in my classroom. According to Stacey & MacGregor (1997), these errors have to do with the stages of cognitive growth required for an individual to progress from concrete to abstract reasoning.

CONCEPTUAL FRAMEWORK

My research project was informed by Kuchemann's (1981) learner interpretation of letters in algebra. According to Kuchemann learners interpret letters in six different ways: letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter as generalized number and letter used as a variable. He further highlights that ascribing different meaning to letters by learners in algebra determines the difficulty of a problem and the extent to which a learner would engage with algebraic problems. In this paper, I focus only on 'letter evaluated', 'letter not used', 'letter used as an object' and 'letter used as an unknown'. The description of these letter interpretations is given briefly below, making use of examples drawn from the literature.

Letter evaluated: It refers to problems that require learners to find the value of an unknown without actually operating on that specific unknown. For example finding the value of $y = x - 3$ if $x = 6$ or calculating the value of p if $p + 2 = 7$. In the first example, learners could evaluate the expression by substituting a given value for x to obtain the value of y. In the second example, learners might view a letter as missing information and this can lead learners to solve the problem by inspection. Booth (1999) refers to this way of solving as an informal method, which might lead to some learners failing to solve similar problems that would typically be solved by transposition or by working with additive identities, such as "find the value of p if $p + 2 = 7 + 2p$". Of course it is also possible, but unlikely, that a learner could solve the above equation by reasoning that $p + 2 = 5 + 2 + p + p$ and concluding that $p = -5$.

Letter not used: In some questions, learners can succeed without actually using the letter, hence the description 'letter not used'. In this case the letter is replaced by a given value. For example, "calculate the value of $2(x + y) - 4 \ if \ x + y = 15$". In such questions learners calculate an answer by substituting the given value. It is noted that learners acknowledge the existence of the variables in a question but do not make sense of them since the answer is a numeric value. Booth (1999) sees such problems as arithmetic since the aim is to find a numeric answer.

Letter used as an object: A variable is treated as shorthand for an object where it represents a length or size of a figure. For example, calculate the perimeter of a shape with equal length of 3 units if there are *n* number of sides Again, it could be regarded as an object when matching or grouping the like-terms. For example, simplify $2b - y + 3b$. In this instance, the meaning of a variable is reduced from abstract to a concrete situation. According to Booth (1999), the unknown tends to have a different meaning when learners view 5*b* as 5 bananas instead of 5 times the number of bananas.

Letter as a specific unknown: It refers to learners viewing a letter as an unknown such that they accept an algebraic expression as an answer. For example, find the product of $3x(2x + 4)$ or add 7 to$x + 1$. In this instance, the letter is considered as having a specific value, which is not known at that time. Hence, the final answer would be in terms of a variable.

The literature review and conceptual framework guided me to understand the errors and misconceptions that learners produce and gave me an indication of how learners interpret letters.

METHODS AND PARTICIPANTS

The participants in the study were Grade 9 learners with ages ranging between 13 and 16 years. I analysed 30 Grade 9 test scripts collected in October 2011 by the Wits Maths Connect Secondary (WMCS) project at a school in the project, to understand errors that learners produce in algebra. After noticing the common errors I developed task-based interviews which I administered with six Grade 9 learners in May 2012. The criterion for selecting these learners was based on their performance in 2012 term 1 mathematics results. Learners who obtained between 45% and 60% qualified to participate in the study.

DATA ANALYSIS AND DISCUSSION OF FINDINGS

As I mentioned, I analysed 30 scripts and conducted task-based interviews with 6 learners, I discuss these two separately and provide some samples of learners' responses.

Documentary analysis of test scripts

The analysis of this data gave me insight into the kinds of errors, mistakes and misconceptions that learners are likely to produce in introductory algebra. Furthermore, it assisted me to develop the instrument for the task-based interview. In one of the questions, learners were asked to "Multiply n+5 by 4". This question was aimed at exploring the use of brackets when multiplying an algebraic expression with two terms. All learners responded to this question and the range of their responses is illustrated in figure 1 below. About 10% of learners responded correctly (4n+20) suggesting that they interpreted the *letter as a specific unknown* (Kuchemann, 1981). Although 7% of learners wrote a correct step, 4(n+5), we cannot assume that they would be able to execute multiplication correctly. About 83% of learners responded incorrectly by giving responses such as n+20, 20n, 4n+5, 20, 24, 45 and $4n^5$. Therefore, this question was one of the areas of focus for interviews with 2012 grade 9 learners.

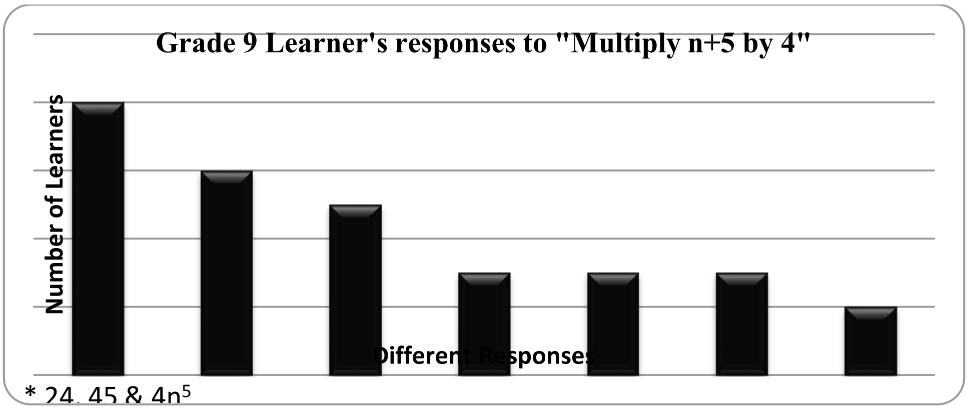
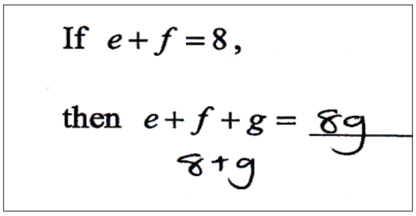
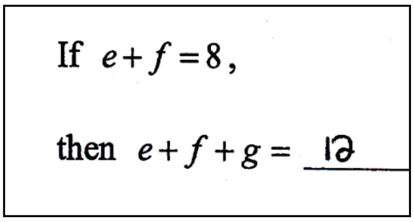


Figure 1: The Bar graph of different learners' responses

In another question learners were asked to “Evaluate e+f+g if e+f = 8”. Of interest here was how learner would deal with the letter *g*. Approximately 17% of learners produced correct responses while a larger proportion 83% of learners produced incorrect responses. Examples of incorrect responses are drawn from Learner 8 and Learner 28 and I present them below.



Learner 8’s response



Learner 28’s response

The range of learners’ responses is illustrated in figure 2. This question was followed up in the task-based interview.

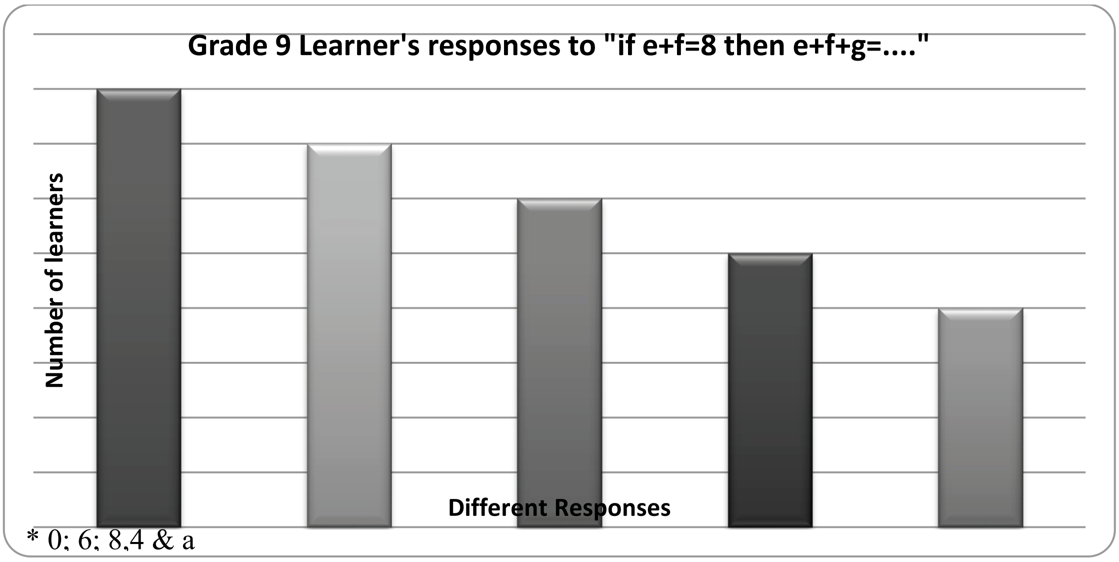


Figure 2: The Bar graph of learners' responses

Task-based interview analysis

A summary of task-based interview analysis is presented in Table 1. Column 1 contains each of the tasks given to the 6 learners during the interview. Column 2 contains the frequencies of correct responses as well as their percentages. Column 3 consists of the frequencies of incorrect responses together with their percentages. Column 4 contains the instances of common errors that were present in learners' responses (for detailed learners' responses see Appendix 1).

Task	*Correct response*	*Incorrect response*	*Nature of Common errors as noted in learners' incorrect responses*
1. *Add 5 to 3x*	*2 (33%)*	*4 (67%)*	• *Conjoined the terms*
2. *Simplify: 3x + 2 + x*	*2 (33%)*	*4 (67%)*	• *Conjoined the terms* • *Interference of new knowledge*
3. *Multiply 3x + 1 by 5*	*3 (50%)*	*3 (50%)*	• *Ignored brackets* • *Conjoined the terms*
4. *Multiply 3x + 1 by x*	*1 (17%)*	*5 (83%)*	• *Ignored brackets* • *Conjoined the terms* • *Interference of new knowledge*
5. *Simplify: 2 (3x + 1)*	*3 (50%)*	*3 (50%)*	• *Conjoined the terms* • *Ignored brackets*
6. *If x + y = 10, then x + y + z =*	*1 (17%)*	*5 (83%)*	• *Conjoined the terms* • *Substituted the letter with numeric value.*

Table 1: Summary of six learners' responses towards the interview task

Task instruction contributed to learner errors

It appears that the task instructions in the interview contributed to some of the errors learners made. This is clearly demonstrated in question 1, where I asked learners to "Add 5 to 3*x*". In Booth's (1999) terms, they interpreted addition as an "action symbol" instead of seeing it as part of the solution, which gave rise to conjoining of terms. Booth (1999) advocates that learners tend to simplify an algebraic solution to a single term. The extract below indicates the learner's explanation concerning the instructions to the questions.

Excerpt 1: Learner A and Learner B's explanations

The learners' response is given in the appendix.

Researcher:	Now I am looking at number 3, 4, 5 and looking at your answers and comparing with 1 & 2. In 1 & 2 you only have one answer but why here (pointing at 3, 4 & 5) you do not have one answer?
Learner A:	Because madam here (referring to number 1 & 2) they said add and here (referring to number 3, 4 &5) they said I must multiply.
Researcher:	Oh! Because in number one you were adding and here, you are multiplying.
Learner A:	Yes, madam.
Researcher:	Why did you not add 15x and 5? (referring to number 3)
Learner A:	Because they are not like-terms.
Researcher:	What about 5 and 3x (referring to number 1)?
Learner A:	Because in number one they said add.
Researcher:	Oh! Because of the instruction.
Learner A:	Yes, madam I added 5 and 3x.
Learner B:	In number one, they said add 5 to 3x. Because they said **add,** me I also added but I know that they are not the like-terms. I understand that they said add that is why I added.

Excerpt 1 indicated that the learner's experience of working with numbers prompted them to treat the instruction as in numeric tasks for example, add 5 to 3. Although these learners knew that 5 and $3x$ were not like-terms but because of their experience with arithmetic context they conjoined the two terms. Again, the instruction 'simplify' in question 2 prompted learners to write their answer in a simplest form of a single term.

Interference of new knowledge gave rise to errors

Learning new concepts in algebra appears to interfere with previous knowledge (Olivier, 1989; MacGregor & Stacey, 1997 & 1999). This was demonstrated by Learner A, when simplifying algebraic expression in question 2. Although Learner A mentioned grouping of like-terms, it is apparent that s/he treated the algebraic expression as an equation. See the extract below.

Excerpt 2: Learner A's explanation

Learner A:	They said 3x+2+x then I said e… when x comes between 3x and 2 it changes the sign to negative x, then I said 3x-x+2, then I got the answer for 3x-x which is 2x and I left the 2 there, then I said 2x+2 which gave me 4x.
Researcher:	Let us go back to where you added the like-terms. You said when x moves closer to 3x it changes the sign. Why?
Learner A:	Because that is how my teacher taught me that when x e… when the equation moves to the other side it changes the sign.
Researcher:	Oh! When it is an equation. Is this an equation?
Learner A:	Yes, madam.
Researcher:	Why do you say it is an equation?
Learner A:	Because madam it has the variables.

Similarly, Learner E demonstrated when multiplying $3x+1$ by x that the knowledge of exponents interfered which gave rise to errors. Below I provide the extract of the learner's explanation.

Excerpt 3: Learner E's explanation

Learner E:	They said we must multiply 3x+1 by x, so I did the same thing as in number 3 I put 3x+1 in brackets and x outside the brackets then multiplied 3x by x which is $3x^2$ and x times 1 which is x then x has the power of one. I said 3x and added the exponent 2 plus 1 which gave me $3x^3$.
Researcher:	(the interviewer did not follow the $3x^3$ since it was outside the scope of the interview).

In excerpt 2, it is clear that the learner could not differentiate between an expression and an equation. This could be the result of the new knowledge of equations which is not deeply rooted (Olivier, 1989). Again, although the learner in excerpt 3 was doing well with other tasks, the errors she produced may have been a result of applying her new knowledge of exponential laws. This shows the diversity of errors produced in introductory algebra. Therefore, the interference of new knowledge portrayed the complexity of working with algebraic problems.

Ignoring a letter gave rise to errors

It was clear that some learners isolate the letter as they continue with some aspects of the procedure. In Kuchemann's (1981) terms this is considered as ignoring the letter. This is clearly demonstrated by learner C who said, "5 plus 3 equal to 8 then I took the x and put it next to 8 to get $8x$". This learner had definitely ignored the letter while doing calculations and attached it in the final response. Below I present the extract of Learner C for question 1, 2 and 3.

Excerpt 4: Learner C's explanation

Learner C: They said add 5 to 3x so I said 5 plus 3 equal to 8 then after that I took the x and put it next to 8 to get 8x.

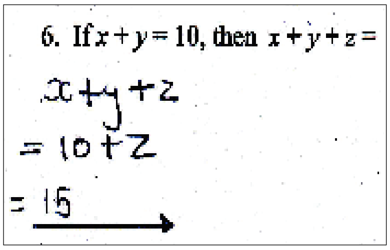
Learner C: They said simplify, is it that there is 3x+2+x, so I said 3 plus 2 equal to 5 then I took the x and put it.

Learner C: They say multiply 3x +1 by x, so here I did the same thing I took 3 and 1 and added together and got 4 then I took the x put it next to 4 and got 4x.

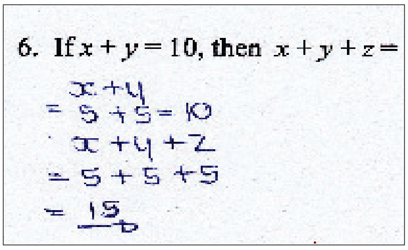
In excerpt 4, it is clear that the learner did not recognize the letter *x* in question 2 and 3. Hence, s/he chose to operate with numeric values. Again learners' experience with numeric context hinders them from dealing with algebraic context. Therefore, this learner reasoned from the numeric perspective which prompted him/her to view the letter as an object.

Replacing a letter with numeric value gave rise to errors

Learners tend to assign a numeric value for letters when simplifying algebraic expressions. According to Christou *et al.*, the tendency of learners "to use their prior experience with numbers in the context of arithmetic" (2007, p. 289) interferes with the interpretation of letters in algebra. This was demonstrated by Learner D and E when they used 5 to replace *x, y and z* (see their response below).



Learner D's response



Learner E's response

These two learners had similar explanations. Below I provide the extract of Learner D's explanation for the response.

Excerpt 5: Learner D's explanation

Learner D: So by number six they say if x+y=10 then x+y+z is equal to what? So I said x+y+z = 10 +z which is the ten I got from x+y then equal to 15, because each number is five because if you say 5+5=10 so I thought maybe they said 5+5 so I gave the z also a five which gave me 15.
Researcher: If you say, 5+5 is equal to 10. Are those the only numbers that can give us 10?
Learner D: No madam
Researcher: Ok, give me a set of two numbers that add up to ten.
Learner D: Eight plus two.
Researcher: Why did you not use eight and two?
Learner D: I gave each the equal amount of number.

Excerpt 5 indicated that the learner could justify his/her strategy to assign equal value for *x, y* and *z.* These learners evaluated the letter by substituting but didn't operate on the letters as unknowns.

CONCLUSION

The research seeks to explore learner thinking underlying explanations of their errors in introductory algebra. The findings indicated that the diversity of errors produced by learners are influenced by to task instruction, new knowledge, ignoring the letter and replacing letter with numeric value as well as other factors. This clearly indicated the complexity of dealing with algebraic expressions. It was noted from the excerpts that learners who seemed to understand like-terms produced errors as a result of task instructions and interference of new knowledge.

Listening to these learners' explanation made me realize that learners grasp information in different ways and respond according to what they have learnt. For example, the learner that said an expression is an equation because it has variables is partly correct but needs to extend that knowledge to observe that an equal sign plays an important role in his definition. This suggests that many errors result from knowledge that is used inappropriately such as over-generalising from instances where it may be appropriate to instances where it does not apply. Moreover, this research project has influenced my practice as a mathematics teacher to be more aware of errors, mistakes and misconceptions that are likely to occur in algebra. I have learnt to accept and appreciate these errors in my classroom and constantly address them to alert learners.

I would conclude by giving teachers a word of advice that these errors do happen in our classroom, however the extent in which one deals with them will determine the learners' conceptual understanding in algebra.

ACKNOWLEDGEMENT

I would like to acknowledge Wits Maths Connect Secondary project for granting me permission to use its data in this paper. Being part of the Transition Maths 1 course illuminated so many things that were happening in my classroom which I could not deal with. Again, thank you WMCS for providing teachers opportunities to participate in professional development. I would also like to thank Shadrack Moalosi from WMCS for his help in writing this paper.

APPENDIX 1: 2012 GRADE 9 RESPONSES ON THE TASK-BASED INTERVIEW

Learners	***Learners' responses to the tasks***					
	1 *Add 5 to 3x*	*2* *Simplify:* $3x + 2 + x$	*3* *Multiply* $3x + 1$ *by 5*	*4* *Multiply* $3x + 1$ *by x*	*5* *Simplify:* $2(3x + 1)$	*6* *If* $x + y = 10$, *then* $x + y + z =$
A	$5+3x$ $= 8x$	$3x-x+2$ $= 2x+2$ $= 4x$	$5(3x+1)$ $=15x+5$	$x(3x+1)$ $=3x^2+x$	$= 6x+2$	$= 10+z$ $=10z$
B	$5+3x$ $= 8x$	$3x+3x$ $=6x$	$3x\times6$ $=18x$	$3x+2x$ $=5x$	$6x+2$ $=8x$	$=10$
C	$8x$	$5x$	$=4x\times5$ $=20x$	$=4x\times x$ $=4x^2$	$2\times3x+1$ $=7x$	$10z$
D	$5+3x$ $= 8x$	$=5x+x$ $=6x$	$3x+1\times5$ $=4x\times5$ $=20x$	$3x+1\times x$ $=4x\times x$ $=4x^2$	$6x+2$ $=8x$	$10+z$ $=15$
E	$5+3x$ $=3x+5$	$=3x+x+2$ $=4x+2$	$5(3x+1)$ $=15x+5$	$x(3x+1)$ $=3x^2+x$ $=3x^{2+1}$ $=3x^3$	$=6x+2$	$x+y=5+5=10$ $x+y+z=5+5+5$ $=15$
F	$3x+5$	$3x+x+2$ $=4x+2$	$5(3x+1)$ $=15x+5$	$x(3x+1)$ $=3x+x$ $=4x$	$6x+2$	$10+z$

REFERENCES

Booth, L. (1999). *Children's difficulties in beginning algebra.* In B. Moses (Ed.), Algebraic Thinking, Grades K-12 (pp. 299–307). Reston, VA: NCTM.

Christou, K.P., Vosniadou, S. & Vamvakoussi, X. (2007). *Students' interpretation of literal symbols in algebra,* Chapter 19, 285-299. Reframing the Conceptual Change Approach in Learning and Instructions: Elsevier Ltd.

Creswell, J.W. (1998). *Qualitative inquiry and research design: Choosing among five traditions.* Thousand Oaks London: SAGE Publishers, International Educational & Professional Publisher.

Department of Basic Education (DBE), (2011). Report on the qualitative analysis of ANA results, Department of Education: South Africa, Pretoria.

Department of Education (DoE), (2002). Revised National Curriculum Statement for Mathematics, Department of Education: South Africa, Pretoria.

Kuchemann, D. (1981). Algebra in K. Hart (ed.). *Children's understanding of mathematics*, 11-16. London: John Murray, pp 102-119.

MacGregor, S. & Stacey, K. (1997). *Students' understanding of algebraic notation*. Educational Studies in Mathematics, 33, 1-19. Kluwer Academic Publishers.

Olivier, A. (1989). *Handling pupils' misconceptions*. Presidential address delivered at the Thirteenth National Convention on Mathematics, Physics and Biology Education, Pretoria, 3-7 July.

Simkins, C. (2013). Performance in the South African educational system: What do we know? Johannesburg: CDE.

Stacey, K. & MacGregor, M. (1999). Ideas *about symbolism that students bring to algebra.* In B. Moses (Ed.), Algebraic Thinking Grades K-12 (pp. 308-312). Reston, VA: NCTM.

Stacey, K., & MacGregor, M. (1994). *Algebraic Sums and Products: Students' Concepts and Symbolism*. In da Ponte, J.P. & Matos, J.F. (eds.) Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education (PME XVIII) (Vol.III): Department of Education, University of Lisbon, Portugal, 289-296.

Vermeulen, N. (2007). *Does Curriculum 2005 promote successful learning of elementary algebra?* Pythagoras 66, 14-33.